ON A RIGIDITY RESULT FOR KOLMOGOROV-TYPE OPERATORS

SU UN RISULTATO DI RIGIDITÀ PER OPERATORI DI TIPO KOLMOGOROV

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ABSTRACT. Let D be a bounded open subset of $\mathbb{R}^N$ and let z_0 be a point of D. Assume that the Newtonian potential of D is proportional outside D to the potential of a mass concentrated at z_0. Then D is a Euclidean ball centred at z_0. This theorem, proved by Aharonov, Schiffer and Zalcman in 1981, was extended to the caloric setting by Suzuki and Watson in 2001. In this note, we extend the Suzuki–Watson Theorem to a class of hypoellliptic operators of Kolmogorov-type.

SUNTO. Sia D un sottoinsieme aperto e limitato di $\mathbb{R}^N$ e sia z_0 un punto D. Assumiamo che il potenziale Newtoniano di D sia proporzionale fuori da D al potenziale di una massa concentrata in z_0. Allora D è una palla Euclidea centrata in z_0. Questo teorema, provato da Aharonov, Schiffer and Zalcman nel 1981, fu esteso all'ambiente calorico da Suzuki e Watson nel 2001. In questa nota estendiamo il Teorema di Suzuki e Watson a una classe di operatori ipoellittici di tipo Kolmogorov.

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1. Introduction

Let us recall some classic results. The harmonic functions, i.e., the solutions to the Laplace equation

$$\Delta u := \sum_{j=1}^{N} \frac{\partial^2 u}{\partial x_j^2} = 0$$

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in some open set $O \subset \mathbb{R}^N$, satisfy the Gauss Mean Value property

$$u(x_0) = M_r(u)(x_0) := \frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} u(\xi)\ d\xi,$$

for every Euclidean ball $B(x_0,r)$, centred at x_0 with radius r, with the closure $\overline{B(x_0,r)}$ contained in O.

The average operators on the Euclidean balls actually characterize the harmonic functions in the following sense. If u is a continuous function in O satisfying the Gauss Mean Value property for every Euclidean ball with its closure contained in O, then u is smooth and harmonic in O.

Furthermore, the Euclidean balls are the only bounded open sets characterizing the harmonic functions. More precisely,

Theorem A. *Let $D \subset \mathbb{R}^N$ be a bounded open set and let $x_0 \in D$. Suppose*

$$u(x_0) = \frac{1}{|D|} \int_D u(\xi) d\xi$$

for every integrable harmonic function u in D. Then D has to be a Euclidean ball centred at x_0.

This theorem, proved by Kuran in 1974, can be also deduced from the following spherical symmetry result obtained in 1981 by Aharonov, Schiffer and Zalcman[1].

Theorem B. *Let $D \subset \mathbb{R}^N$ be a bounded open set and let $x_0 \in D$. Assume there exists a real constant $c > 0$ such that*

$$\frac{1}{c} \int_D \frac{1}{|\xi - x|^{N-2}}\ d\xi \ = \ \frac{1}{|x_0 - x|^{N-2}} \quad \textit{for every } x \in \mathbb{R}^N \setminus D.$$

Then D is a Euclidean ball centred at x_0 and c is the Lebesgue measure of D.

In order to deduce the Kuran Theorem we just need to recall that, denoting by K the fundamental solution of the Laplace operator at the origin, the function $\xi \longmapsto K(\xi - x) = \frac{1}{|\xi - x|^{N-2}}$ is harmonic and integrable in D for every $x \in \mathbb{R}^N \setminus D$.

[1]Actually Aharonov, Schiffer and Zalcman use Kuran Theorem in the proof of Theorem B. A direct proof of Theorem B that does not require Theorem A can be found in [2].

Let us mention that the Aharonov, Schiffer and Zalcman Theorem was obtained to solve a physical problem posed by Rubel: if the Newtonian potential of D is proportional outside D to the potential of a mass concentrated at a single point, then D must be spherical.

In 2001 Suzuki and Watson extended this theorem to the caloric setting. To state their result we need to introduce some notation.

First of all, we denote by

$$\mathcal{H} = \Delta - \partial_t$$

the heat operator in $\mathbb{R}^{N+1} = \mathbb{R}^N \times \mathbb{R} \ni z = (x,t)$. G will stand for the Gauss–Weierstrass kernel, i.e., the fundamental solution at the origin to $\mathcal{H}$:

$$G(x,t) = \frac{1}{(4\pi t)^{\frac{N}{2}}} \exp\left(-\frac{|x|^2}{4t}\right) \text{ if } t>0, \qquad G(x,t) = 0 \text{ if } \ t \le 0.$$

We define the heat balls or caloric balls centred at $z_0 \in \mathbb{R}^{N+1}$ with radius $r > 0$ as follows:

$$\Omega_r(z_0) = \left\{ z \in \mathbb{R}^{N+1} : G(z_0 - z) > \frac{1}{r} \right\}.$$

We remark that, since the fundamental solution has the support in a halfspace, the "centre" z_0 is actually a point of the boundary of the caloric balls.

The caloric functions, i.e., the solutions to the heat equation

$$\mathcal{H}u = 0$$

in some open set $O \subset \mathbb{R}^{N+1}$, satisfy an analogous of the Gauss Mean Value property. The history of the Mean Value Theorem for caloric functions started with a paper by Pini dated 1951 ([11]) and the following formula that characterizes the caloric functions was discovered by Watson in 1973 ([13]).

Theorem C. *Let $u \in C(O, \mathbb{R})$. Then, $u \in C^\infty(O,\mathbb{R})$ and $\mathcal{H}u = 0$ in O if and only if*

$$u(z_0) = M_r(u)(z_0) : \frac{1}{r} \int_{\Omega_r(z_0)} u(\zeta) W(z_0 - \zeta)\, d\zeta,$$

for every heat ball $\Omega_r(z_0)$ such that $\overline{\Omega_r(z_0)} \subseteq O$.

The kernel W is given by

$$W(\zeta) = W(\xi, \tau) = \frac{|\xi|^2}{4\tau^2}.$$

As a consequence, since $\zeta \longmapsto G(\zeta - z)$ is caloric in $\Omega_r(z_0)$ for every $z \notin \Omega_r(z_0)$, one has

$$G(z_0 - z) = \frac{1}{r} \int_{\Omega_r(z_0)} G(\zeta - z) W(z_0 - \zeta) d\zeta,$$

for every heat ball $\Omega_r(z_0)$, $\overline{\Omega_r(z_0)} \subseteq O$, and for every $z \notin \Omega_r(z_0)$.

Suzuki and Watson extended Aharonov, Schiffer and Zalcman's Theorem to the caloric setting proving that this identity is actually a rigidity property of the heat balls. We can now finally state their result:

Theorem D. *Let $z_0 = (x_0, t_0) \in \mathbb{R}^{N+1}$ and let D be a bounded open set of $\mathbb{R}^{N+1}$. Assume that for a suitable constant $c > 0$,*

$$\int_D G(\zeta - z) W(z_0 - \zeta) \ d\zeta = c\, G(z_o - z) \quad \forall\, z \notin D.$$

Assume also that

$$\text{(SW)} \qquad \zeta \longmapsto (\mathbb{1}_D - \mathbb{1}_{\Omega_c(z_0)})(\zeta) W(z_0 - \zeta) \in L^p \text{ for some } p > \frac{n}{2} + 1,$$

then,

$$D = \Omega_c(z_0).$$

Here and henceforth, $\mathbb{1}_E$ denotes the characteristic function: $\mathbb{1}_E(x) = 1$ if $x \in E$, $\mathbb{1}_E(x) = 0$ otherwise.

We remark that condition (SW) takes the place of the condition that the point x_0 is in the interior of D in the harmonic case. To use Suzuki and Watson's words, its meaning is that D and $\Omega_c(z_0)$ are "indistinguishable in the vicinity of z_0". It can be replaced, e.g., by the stronger conditions:

(KLT) *(i) there exists a neighborhood V of z_0 s. t. $\Omega_r(z_0) \cap V = D \cap V$;*
(ii) $\overline{D} \setminus \{z_0\} \subset \mathbb{R}^N \times]-\infty, t_0[$.

Very recently, we extended with Lanconelli the Suzuki–Watson Theorem to the $\mathcal{L}$-setting, where $\mathcal{L}$ is a Kolmogorov-type operator

$$\mathcal{L} = \operatorname{div}(A\nabla) + \langle Bx, \nabla\rangle - \partial_t, \text{ in } \mathbb{R}^{N+1} = \mathbb{R}^N_x \times \mathbb{R}_t$$

where $A = (a_{ij})_{i,j=1,\dots,N}$ and $B = (b_{ij})_{i,j=1,\dots,N}$ are $N \times N$ matrices with real constant coefficients, A is symmetric and non-negative definite and A and B satisfy additional suitable conditions in order that the operator $\mathcal{L}$ is hypoelliptic and homogeneous.

We will describe this class of operators and we will state our theorem in detail in the next Sections 2 and 3.

In 2014, together with Lanconelli and Tralli, in [6], we had already extended this last rigidity theorem to a more general class of second order hypoelliptic operators containing in particular the Kolmogorov-type operators of above, but under the hypotheses (KLT).

The new proof of our main rigidity result does not use any symmetry technique: it closely follows the lines introduced by Cupini and Lanconelli in the paper [2], where harmonic characterizations of the Euclidean balls are proved by only using ideas and results from "elliptic" Potential Analysis. We extend the method in [2] to the setting of the Kolmogorov operators by exploiting the more general Potential Analysis for evolution linear second order hypoelliptic PDEs. We will give the main ideas of the proof in Section 4. For the complete proofs we refer to the work [5], on which this seminar is based.

2. Our class of operators

The operators we are dealing with are Kolmogorov-type operators in $\mathbb{R}^{N+1}$ of the following type

$$\mathcal{L} = \operatorname{div}(A\nabla) + \langle Bx, \nabla\rangle - \partial_t, \tag{1}$$

where div, ∇, $\langle\cdot,\cdot\rangle$ stand respectively for the divergence, the Euclidean gradient and the inner product in $\mathbb{R}^N$. $A = (a_{ij})_{i,j=1,\dots,N}$ and $B = (b_{ij})_{i,j=1,\dots,N}$ are $N \times N$ matrices with real constant coefficients, A is symmetric and non-negative definite. We suppose the operator $\mathcal{L}$ to be hypoelliptic and homogeneous of degree two with respect to a group of dilations.

We know (see [10], see also [3]) that these properties are guaranteed, e.g., by the following conditions on the matrices A and B. The matrix A has to take the following block structure:

$$A = \begin{bmatrix} A_0 & 0 \\ 0 & 0 \end{bmatrix},$$

for some $p_0 \times p_0$ symmetric and strictly positive definite matrix A_0, $p_0 \leq N$. Moreover, if $p_0 < N$, i.e., if $\mathcal{L}$ is a degenerate elliptic-parabolic operator, the matrix B has to be written as follows

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ B_1 & 0 & 0 & 0 & 0 \\ 0 & B_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & B_n & 0 \end{bmatrix},$$

where B_j is a $p_j \times p_{j-1}$ matrix with maximum rank p_j; $j = 1, 2, ..., n$, $p_0 \geq p_1 \geq ... \geq p_n \geq 1$ and $p_0 + p_1 + ... + p_n = N$.

In 1994 Lanconelli and Polidoro in [10] studied the operators $\mathcal{L}$ in (1) and they proved their left translation invariance with respect to the Lie group $\mathbb{K} = (\mathbb{R}^{N+1}, \cdot, \delta_r)$ with composition law

$$(x, t) \cdot (x', t') = (x' + E(t')x, t + t'),$$

where E represents the matrix

$$E(s) := e^{-sB}, \quad s \in \mathbb{R}.$$

Furthermore, they proved their homogeneity with respect to the group of dilations

$$\delta_\lambda : \mathbb{R}^{N+1} \longrightarrow \mathbb{R}^{N+1}, \quad \delta_\lambda(x, t) := (\lambda x^{(p_0)}, \lambda^3 x^{(p_1)}, \dots, \lambda^{2n+1} x^{(p_n)}, \lambda^2 t);$$

where $x^{(p_i)} \in \mathbb{R}^{p_i}$, $i = 0, \dots, n$, and $\lambda > 0$. Since the operator is hypoelliptic, the matrix

$$C(t) = \int_0^t E(s) A E^T(s) \, ds$$

is strictly positive definite for every $t > 0$. An explicit fundamental solution for $\mathcal{L}$ is given by

$$\Gamma(z,\zeta) := \gamma\left(\zeta^{-1} \circ z\right) \text{ for } z, \zeta \in \mathbb{R}^{n+1},$$

where $\zeta^{-1} = (\xi, \tau)^{-1} = (-E(-\tau)\xi, -\tau)$ is the opposite of ζ with respect to the composition law and

$$\gamma(z) = \gamma(x,t) := \begin{cases} 0 & \text{for } t \le 0 \\ \frac{(4\pi)^{-N/2}}{\sqrt{\det C(t)}} \exp\left(-\frac{1}{4}\left\langle C^{-1}(t)\, x, x\right\rangle\right) & \text{for } t > 0 \end{cases}.$$

The function γ is the fundamental solution of $\mathcal{L}$ with pole at the origin and is δ_λ-homogeneous of degree $2 - Q$, i.e.,

$$\gamma(\delta_\lambda(z)) = \lambda^{2-Q}\gamma(z) \quad \forall\, z \in \mathbb{R}^{N+1},\ \forall\, \lambda > 0.$$

The natural number $Q = p_0 + 3p_1 + \ldots + (2r+1)p_r + 2$ is the homogeneous dimension related to the family of dilations δ_λ.

A celebrated example of operator belonging to this class is the kinetic Kolmogorov operator. Let $\mathbb{I}_n$ be the identity $n \times n$ matrix, let N be equal to $2n$ and let the matrices A and B be of the type

$$A = \begin{pmatrix} \mathbb{I}_n & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 0 & 0 \\ \mathbb{I}_n & 0 \end{pmatrix},$$

one recovers the prototype $\mathcal{K}$ of the operators introduced in 1934 by Kolmogorov in studying diffusion phenomena from a probabilistic point of view:

$$\mathcal{K} = \operatorname{div}(A\nabla) + \langle Bx, \nabla\rangle - \partial_t = \sum_{i=1}^{n} \partial_{x_i}^2 + \sum_{i=1}^{n} x_i \partial_{x_{n+i}} - \partial_t \ \text{ in } \mathbb{R}^{2n+1}.$$

In the model of Kolmogorov the positive solutions of $\mathcal{K}u = 0$ are probability densities of a system having $2n$ *degrees of freedom*. The $2n$ dimensional space is the phase space: $(x_1, \ldots, x_n)$ is the velocity vector and $(x_{n+1}, \ldots, x_{2n})$ is the vector of the positions of the system. In this case:

$$E(s) = e^{-sB} = \mathbb{I}_{2n} - sB$$

$$E(s)\ A\ E(s)^{T} = \begin{pmatrix} \mathbb{I}_n & \mathbb{0}_n \\ -s\mathbb{I}_n & \mathbb{I}_n \end{pmatrix} \begin{pmatrix} \mathbb{I}_n & \mathbb{0}_n \\ \mathbb{0}_n & \mathbb{0}_n \end{pmatrix} \begin{pmatrix} \mathbb{I}_n & -s\mathbb{I}_n \\ \mathbb{0}_n & \mathbb{I}_n \end{pmatrix},$$

so that,

$$E(s)\ A\ E(s)^{T} = \begin{pmatrix} \mathbb{I}_n & -s\mathbb{I}_n \\ -s\mathbb{I}_n & s^2\mathbb{I}_n \end{pmatrix}$$

and

$$C(t) = \int_0^t E(s)\ A\ E(s)^{T}\ ds = \begin{pmatrix} t\mathbb{I}_n & -\frac{t^2}{2}\mathbb{I}_n \\ -\frac{t^2}{2}\mathbb{I}_n & \frac{t^3}{3}\mathbb{I}_n \end{pmatrix}.$$

Then, since

$$\det C(t) = \left(\frac{1}{12}\right)^n t^{4n} > 0, \quad \forall t > 0,$$

C is strictly positive definite for every $t > 0$.

The composition law of the group $\mathbb{K}$ related to the Kolmogorov operator can be explicitly written. Indeed we have just observed that

$$E(t) = \mathbb{I}_{2n} - tB = \begin{pmatrix} \mathbb{I}_n & \mathbb{0}_n \\ -t\mathbb{I}_n & \mathbb{I}_n \end{pmatrix}.$$

As a consequence,

$$(z \circ \zeta) = (x,t) \circ (\xi,\tau) = (\xi + E(\tau)x, t+\tau) = (\xi' + x', \xi'' + x'' - \tau x', t + \tau),$$

where $x = (x', x'')$ and $\xi = (\xi', \xi'')$, $x', x'', \xi', \xi'' \in \mathbb{R}^n$. Moreover, K is homogeneous of degree two with respect to the dilations

$$\delta_\lambda : \mathbb{R}^{2n+1} \longrightarrow \mathbb{R}^{2n+1}$$

$$\delta_\lambda(x,t) = \delta_\lambda(x', x'', t) = (\lambda x', \lambda^3 x'', \lambda^2 t).$$

The homogeneous dimension of this group is $Q = n + 3n + 2 = 4n + 2$.

3. Main results

As in the case of the heat operator, we will define the $\mathcal{L}$-balls centred at z_0 as superlevel sets of the fundamental solution with pole at z_0. So, for every $z_0 \in \mathbb{R}^{N+1}$ and $r > 0$ we call *$\mathcal{L}$-ball with center z_0 and radius $r > 0$* the following open set

$$\Omega_r(z_0) = \left\{ z \in \mathbb{R}^{N+1} : \Gamma(z_0, z) > \frac{1}{r} \right\}.$$

Over these balls we define the kernel W as follows

$$W(z) = W(x,t) := \frac{\langle AC^{-1}(t)x, C^{-1}(t)x \rangle}{4}.$$

We observe that W is well-defined and moreover is a smooth function which is strictly positive almost everywhere in $\mathbb{R}^{N+1}$. We remark that in the particular case

$$A = \mathbb{I}_N,\ B = \mathbb{0}_N,\ E(t) = \mathbb{I}_N,\ C(t) = t\mathbb{I}_N,$$

then $\mathcal{L}$ becomes the heat operator and W coincides with the Pini–Watson kernel of the heat operator. Also in this setting we have an analogous of the Gauss Mean Value theorem, essentially due to Kupcov (see [9], see also [10], [6, Theorem 1.1]), for the $\mathcal{L}$-harmonic functions in O, i.e., the solutions to $\mathcal{L}u = 0$ in O.

Theorem E. *Let $u \in C(O, \mathbb{R})$. Then, $u \in C^\infty(O, \mathbb{R})$ and $\mathcal{L}u = 0$ in O if and only if*

$$u(z_0) = M_r(u)(z_0) =: \frac{1}{r} \int_{\Omega_r(z_0)} u(\zeta) W(z_0^{-1} \circ \zeta)\ d\zeta,$$

for every $\mathcal{L}$-ball $\Omega_r(z_0)$ such that $\overline{\Omega_r(z_0)} \subseteq O$.

From this Mean Value formula, just proceeding as in the caloric case, one gets

$$\int_{\Omega_r(z_0)} \Gamma(\zeta, z) W(z_0^{-1} \circ \zeta)\ d\zeta = r\Gamma(z_0, z) \quad \forall\, z \notin \Omega_r(z_0).$$

We will prove that this is a rigidity property of the $\mathcal{L}$-balls, extending to the Kolmogorov setting the Suzuki–Watson theorem.

Theorem 3.1. *Let $z_0 \in \mathbb{R}^{N+1}$ and let D be a bounded open subset of $\mathbb{R}^{N+1}$ such that, for a suitable $r > 0$,*

$$\int_D \Gamma(\zeta, z) W(z_0^{-1} \circ \zeta)\ d\zeta = r\Gamma(z_0, z) \quad \forall\, z \in \mathbb{R}^{N+1} \setminus D.$$

If, moreover,

$$\text{(SW')} \qquad \zeta \longmapsto (\mathbb{1}_D - \mathbb{1}_{\Omega_r(z_0)}) W(z_0^{-1} \circ \zeta) \in L^p \text{ for some } p > \frac{Q}{2},$$

then $D = \Omega_r(z_0)$.

As a corollary, for our class of operators we will have in our setting an analogous of the Kuran Theorem for the harmonic functions.

Corollary 3.1. *Let $z_0 \in \mathbb{R}^{N+1}$ and let D be a bounded open subset of $\mathbb{R}^{N+1}$ such that, for a suitable $r > 0$,*

$$u(z_0) = \frac{1}{r} \int_D u(\zeta) W(z_0^{-1} \circ \zeta)\ d\zeta$$

for every non negative function u $\mathcal{L}$-harmonic in an open set containing $D \cup \{z_0\}$. If, moreover, condition (SW′) holds, then $D = \Omega_r(z_0)$.

4. Proof of theorem 3.1: key ideas

We introduce two Radon measures, μ and ν, and the corresponding "Γ-potentials" Γ_μ and Γ_ν. For fixed $z_0 \in \mathbb{R}^{N+1}$ and $r > 0$, we define μ as the Radon measure in $\mathbb{R}^{N+1}$ such that

$$d\mu(\zeta) = \frac{1}{r} \mathbb{1}_{\Omega_r(z_0)}(\zeta) W(z_0^{-1} \circ \zeta)\ d\zeta.$$

From the Mean Value formula,

$$\int_{\Omega_r(z_0)} \Gamma(\zeta, z) W(z_0^{-1} \circ \zeta)\ d\zeta = r\Gamma(z_0, z) \quad \forall\, z \notin \Omega_r(z_0),$$

so for every $z \notin \Omega_r(z_0)$

$$\Gamma_\mu(z) = \int_{\mathbb{R}^{N+1}} \Gamma(z, \zeta)\ d\mu(\zeta) = \frac{1}{r} \int_{\Omega_r(z_0)} \Gamma(\zeta, z) W(z_0^{-1} \circ \zeta)\ d\zeta = \Gamma(z_0, z).$$

Whereas ν, for fixed $z_0 \in \mathbb{R}^{N+1}$ and $r > 0$, is the Radon measure in $\mathbb{R}^{N+1}$ such that

$$d\nu(\zeta) = \frac{1}{r}\mathbb{1}_D(\zeta) W(z_0^{-1}\circ\zeta)\ d\zeta.$$

From the assumption in Theorem 3.1,

$$\int_D \Gamma(\zeta,z) W(z_0^{-1}\circ\zeta)\ d\zeta = r\Gamma(z_0,z) \quad \forall\, z\notin D,$$

so for every $z\notin D$

$$\Gamma_\nu(z) = \int_{\mathbb{R}^{N+1}} \Gamma(z,\zeta)\ d\nu(\zeta) = \frac{1}{r}\int_D \Gamma(\zeta,z) W(z_0^{-1}\circ\zeta)\ d\zeta = \Gamma(z_0,z).$$

To recap, as a consequence of the definitions of μ and ν, we have first of all

$$\mu|_{\Omega_r(z_0)\cap D} = \nu|_{\Omega_r(z_0)\cap D};$$

moreover

$$\Gamma_\mu(z) = \Gamma(z_0,z) \quad \forall\, z\notin \Omega_r(z_0)$$

and

$$\Gamma_\nu(z) = \Gamma(z_0,z) \quad \forall\, z\notin D$$

imply

$$\Gamma_\mu(z) = \Gamma_\nu(z) \quad \forall\, z\in\mathbb{R}^{N+1}\setminus(\Omega_r(z_0)\cup D).$$

We extend the equality of the Γ-potentials to the following set

$$\Gamma_\mu(z) = \Gamma_\nu(z) \quad \forall\, z\in\mathbb{R}^{N+1}\setminus\Omega_r(z_0).$$

To this end, we use some basic notions and results of Potential Theory for $\mathcal{L}$ and for its adjoint $\mathcal{L}^\star$, a strong parabolic maximum principle for the $\mathcal{L}^\star$-subharmonic functions and a propagation theorem of the maxima (see e.g.[7]). Furthermore, we use the assumption (SW$'$) in order to get the continuity of $\Gamma_\mu - \Gamma_\nu$ (this follows from a real analysis convolution result). Then, from

$$\Gamma_\mu(z) = \Gamma_\nu(z) \quad \forall\, z \in \mathbb{R}^{N+1} \setminus \Omega_r(z_0),$$

we obtain that D has to be a subset of Ω. Indeed, being Γ the fundamental solution, $\mathcal{L}^*\Gamma_\mu = -\mu$ and so Γ is $\mathcal{L}^\star$-harmonic outside its support in $\mathbb{R}^{N+1} \setminus \overline{\Omega}$.
As a consequence,

$$\nu = -\mathcal{L}^\star(\Gamma_\nu) = -\mathcal{L}^\star(\Gamma_\mu) = 0 \;\text{ in } \mathbb{R}^{N+1} \setminus \overline{\Omega},$$

i.e.,

$$\nu(\mathbb{R}^{N+1} \setminus \overline{\Omega}) = 0,$$

or, equivalently,

$$\operatorname{supp} \nu \subseteq \overline{\Omega}.$$

Then $\operatorname{supp} \nu = \overline{D} \subseteq \overline{\Omega}$ and $D \subseteq \operatorname{int}(\overline{\Omega})$. On the other hand, as γ is δ_λ-homogeneous of degree $2 - Q$,

$$\operatorname{int}(\overline{\Omega}) = \Omega.$$

Hence,

$$D \subseteq \Omega.$$

The last step is to prove that $D = \Omega$. We argue by contradiction and assume $D \neq \Omega$. In this case, there exists $z \in \Omega$ such that $z \notin D$. As a consequence of a Poisson–Jensen-type formula (see [8, Corollary 3.2]), we get that for every $z \in \Omega$

$$\Gamma(z_0, z) > \Gamma_\mu(z).$$

Moreover, we know that μ and ν coincide on $\Omega \cap D$; so, as $D \subseteq \Omega$, we have

$$\mu|_D = \nu.$$

In addition, since $z \notin D$, from our theorem's assumption,

$$\Gamma_\nu(z) = \Gamma(z_o, z).$$

Combining the above, we have

$$\begin{aligned}\Gamma(z_0,z) &> \Gamma_\mu(z)=\int_\Omega \Gamma(\zeta,z)\ d\mu(\zeta)\\ &\geq \int_D \Gamma(\zeta,z)\ d\mu(\zeta)=\int_D \Gamma(\zeta,z)\ d\nu(\zeta)=\Gamma_\nu(z)=\Gamma(z_o,z),\end{aligned}$$

that is $\Gamma(z_0,z)>\Gamma(z_0,z)$. This contradiction proves $D=\Omega$ and completes the proof of our theorem.

We are currently working on extending this method to more general class of Partial Differential Equations; in particular to the class of the Hörmander evolution operators

$$\sum_{i=1}^{m} X_j^2 + X_0 - \partial_t \ \text{ in } \mathbb{R}^{N+1}$$

left invariant and homogeneous of degree two on a homogeneous Lie group introduced and studied in [4].

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